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Instability of the flow between rotating cylinders: the wide-gap problem

By E. M. SPARROW, W. D. MUNRO AND V. K. JONSSON
University of Minnesota, Minneapolis, Minnesota

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An analytical investigation is carried out to determine the conditions for instability in a viscous fluid contained between rotating coaxial cylinders of arbitrary radius ratio. A solution method is outlined and then applied to cylinders having radius ratios ranging from 0·95 to 0·1. Consideration is given to both cases wherein the cylinders are rotating in the same direction and in opposite directions. Results are reported for the Taylor numbers and wave-numbers which mark the onset of instability. The present results are also employed to delineate the range of applicability of the closed-form instability predictions of Taylor and of Meksyn, which were derived for narrow-gap conditions.

1. Introduction

The stability of the flow contained between two rotating cylinders has been a subject of continuing interest for many years. The background of the problem and most of the major contributions are described in the monumental treatise on hydrodynamic stability of Chandrasekhar (1961). Consequently, the present discussion of the literature will be limited to those matters which bear directly on this investigation.

Consideration is given to a viscous incompressible fluid contained between coaxial cylinders of radii R_1 and $R_2\,(R_2 > R_1)$ which rotate with angular velocities Ω_1 and Ω_2, respectively. There is no axial flow. It has been established that a flow pattern characterised by circular streamlines is stable against small disturbances when the inner cylinder is stationary, i.e. $\Omega_1 = 0$. However, when the inner cylinder is rotating, the aforementioned flow pattern may break down under the action of small disturbances. The specific conditions under which instability sets in have been explored within the framework of small-perturbation hydrodynamic stability theory.

In formulating the stability problem, it has been common to consider the case in which the gap $(R_2 - R_1)$ between the cylinders is small compared with the mean radius. Within this restriction and with the further assumption that the speeds Ω_1 and Ω_2 are of the same sign, Taylor (1923) derived a pair of closed-form expressions for predicting the onset of instability. The first of these is the more frequently quoted and pertains to the case where $(R_2 - R_1)/R_1$ is fully negligible. The second applies 'when $(R_2 - R_1)/R_1$ is small, but not neglected'. Later, in a series of papers which are summarized in his book, Meksyn (1961) attacked the problem by a mathematical method different from that of Taylor. By seeking separate asymptotic solutions for the cases $\Omega_2/\Omega_1 \geqslant 0$ and $\Omega_2/\Omega_1 < 0$, he was able to derive closed-form predictions for the instability condition.

More recently, several aspects of the stability problem for the narrow gap have been reconsidered by Chandrasekhar (1961). Among these, an alternate derivation of Taylor's closed-form instability criterion was devised. Furthermore, a numerical computation involving a trial and error evaluation of a determinant was carried out, thereby providing information on instability both for $\Omega_2/\Omega_1 \geqslant 0$, and $\Omega_2/\Omega_1 \leqslant 0$. In addition to the foregoing, the problem has also been attacked by variational methods, e.g. Di Prima (1960).

When the restriction to narrow gaps is lifted, the difficulties in solving the stability problem are compounded. This may well account, at least in part, for the fact that the wide-gap case has heretofore received only restricted analytical consideration. Within the knowledge of the authors, the only pertinent investigations presently available in the open literature are due to Chandrasekhar (1961), Chandrasekhar & Elbert (1962), Kirchgässner (1961), and Witting (1958). The solutions due to Chandrasekhar involve the use of Bessel series and ultimately lead to the condition that an infinite-order determinant be zero. Kirchgässner's solution method is based on the construction of an appropriate Green's function. The treatment of Witting makes use of an expansion in powers of $\xi = 1-(R_1/R_2)$, wherein ξ is treated as a small parameter. The information relating to the onset of instability that has been provided by the aforementioned references is restricted to a relatively narrow range of the parameters.

The present investigation is concerned with the case of the gap of arbitrary radius ratio, and a direct and relatively simple approach is devised for determining the instability criteria. The method is utilized in computing the stability characteristics of rotating systems with radius ratios R_1/R_2 ranging from 0·95 to 0·1 for both conditions where $\Omega_2/\Omega_1 \geqslant 0$ or $\Omega_2/\Omega_1 \leqslant 0$. The present results are also used in delineating the range of applicability of the aforementioned closed-form predictions of Taylor and Meksyn.

2. The stability problem and its solution

Analytical formulation. The formulation of the stability problem has been carried out by numerous investigators and is also reported in various reference books (e.g. Lin 1955). Consequently, there is no need to repeat the details here. In essence, one begins with the equations of motion and continuity in cylindrical co-ordinates (r, θ, z) for unsteady, incompressible flow. The radial co-ordinate is measured from the axis of the coaxial cylinders, while z lies along the axis. The steady, axi-symmetric solution of these equations is known to be

$$V_r = 0, \quad V_\theta = Ar + B/r, \quad V_z = 0, \tag{1}$$

where

$$A = \frac{\Omega_1(1-\mu\eta^{-2})}{1-\eta^{-2}}, \quad B = \frac{R_1^2\Omega_1(1-\mu)}{1-\eta^2}, \tag{1a}$$

and

$$\mu = \Omega_2/\Omega_1, \quad \eta = R_1/R_2, \tag{2}$$

the latter being the standard notations for the speed ratio and the radius ratio.

For the stability problem, one postulates small perturbations v_r, v_θ, and v_z superposed on the main flow. Then the equations of motion are linearized by discarding products and squares of v_r, v_θ, and v_z. It is further postulated that the perturbation quantities have the form

$$v_r = u(r)\, e^{\sigma t} \cos \lambda z, \quad v_\theta = v(r)\, e^{\sigma t} \cos \lambda z, \quad v_z = w(r)\, e^{\sigma t} \sin \lambda z, \tag{3}$$

where λ is proportional to the wave-number of the disturbance in the z-direction and σ may be regarded as an amplification factor. The expressions in (1) and (3) are introduced into the linearized equations of motion and the continuity equation. After some manipulation, the static pressure p and the amplitudes $u(r)$ and $w(r)$ can be eliminated, leaving a single ordinary differential equation for the amplitude $v(r)$ of the tangential perturbation. To specialize this equation to the study of the onset of instability, one sets $\sigma = 0$ (the marginal stability state is stationary). The execution of the foregoing operations and the introduction of dimensionless variables yields

$$\begin{aligned} &\frac{d^6 v}{d\rho^6} + \frac{3}{\rho}\frac{d^5 v}{d\rho^5} - \left(\frac{6}{\rho^2} + 3\Lambda^2\right)\frac{d^4 v}{d\rho^4} + \left(\frac{12}{\rho^3} - \frac{6\Lambda^2}{\rho}\right)\frac{d^3 v}{d\rho^3} \\ &\quad + \left(-\frac{27}{\rho^4} + \frac{9\Lambda^2}{\rho^2} + 3\Lambda^4\right)\frac{d^2 v}{d\rho^2} + \left(\frac{45}{\rho^5} - \frac{9\Lambda^2}{\rho^3} + \frac{3\Lambda^4}{\rho}\right)\frac{dv}{d\rho} \\ &\quad + \left[-\frac{45}{\rho^6} + \frac{9\Lambda^2}{\rho^4} - \frac{3\Lambda^4}{\rho^2} - \Lambda^6 - 4\Lambda^2 T^* \left(C + \frac{D}{\rho^2}\right)\right] v = 0, \end{aligned} \tag{4}$$

wherein

$$\rho = r/R_2, \tag{5a}$$

$$T^* = \Omega_1^2 R_2^4/\nu^2, \quad \Lambda = \lambda R_2, \tag{5b}$$

$$C = \left[\frac{1-\mu\eta^{-2}}{1-\eta^{-2}}\right]^2, \quad D = \eta^2 \left[\frac{(1-\mu\eta^{-2})(1-\mu)}{(1-\eta^{-2})(1-\eta^2)}\right]. \tag{5c}$$

It is seen that equation (4) is a sixth-order, linear, homogeneous differential equation with variable coefficients. There are four parameters which appear in the equation: the speed ratio μ, the radius ratio η, the dimensionless wave-number Λ, and the grouping $\Omega_1^2 R_2^4/\nu^2$. The latter will be designated here as a Taylor number and is denoted by T^*. In this connexion, it may be noted that there are several groupings in the literature which are referred to as the Taylor number and care must therefore be exercised in comparing results of different investigations.

The boundary conditions for the stability problem require that the perturbation amplitudes u, v, and w be zero on the bounding surfaces $r = R_1$ and $r = R_2$. By applying the equation of continuity and other relationships derivable from the equations of motion, it is possible to recast all of the foregoing into boundary conditions on v and its derivatives. Thus, at $\rho = \eta$,

$$v = 0, \quad \frac{d^2 v}{d\rho^2} + \frac{1}{\eta}\frac{dv}{d\rho} = 0, \quad \frac{d^3 v}{d\rho^3} - \left(\frac{3}{\eta^2} + \Lambda^2\right)\frac{dv}{d\rho} = 0, \tag{6a}$$

and, at $\rho = 1$,

$$v = 0, \quad \frac{d^2 v}{d\rho^2} + \frac{dv}{d\rho} = 0, \quad \frac{d^3 v}{d\rho^3} - (3 + \Lambda^2)\frac{dv}{d\rho} = 0. \tag{6b}$$

Inasmuch as the governing equation and the boundary conditions are homogeneous, it follows that equations (4) and (6) can be regarded as an eigenvalue problem from which the criteria for the onset of instability can be derived. In particular, if one fixes the speed ratio μ and the radius ratio η, then, corresponding to a given Λ, one can determine the value of T^* for which a solution to the eigenvalue problem exists. Further, by varying the assigned value of Λ, one can determine a corresponding range of T^* values for which the solution exists. Experience suggests that, among these admissible T^* values, there is one which is lowest. In other words, below this lowest T^* value, a solution for the perturbation amplitude cannot be found. Thus, the minimum T^* corresponds to the condition at which instability sets in. It is evident that the critical T^* may depend on the assigned values of the speed ratio μ and the radius ratio η.

Whereas the foregoing discussion sets forth the connexion between the solution of equation (4) and the onset of instability, it still remains to describe how solutions of equations (4) and (6) are to be obtained. The solution method will be discussed in the succeeding paragraphs.

The solution method. The essential task is to find a value of T^* which, for prescribed μ, η, and Λ, leads to a solution for v that satisfies the differential equation (4) and the boundary conditions (6). The first step in the solution method is to select a trial value of T^*. Then, corresponding to this T^*, one constructs by numerical means three solutions v_{I}, v_{II}, v_{III} of equation (4) which satisfy the following initial conditions:

(*a*) v_{I} satisfies equation (6*a*), and in addition

$$dv_{\mathrm{I}}/d\rho = 1, \quad d^4v_{\mathrm{I}}/d\rho^4 = d^5v_{\mathrm{I}}/d\rho^5 = 0 \quad \text{at} \quad \rho = \eta.$$

(*b*) v_{II} satisfies equation (6*a*), and in addition $dv_{\mathrm{II}}/d\rho = 0$, $d^4v_{\mathrm{II}}/d\rho^4 = 1$, $d^5v_{\mathrm{II}}/d\rho^5 = 0$ at $\rho = \eta$.

(*c*) v_{III} satisfies equation (6*a*) and in addition $dv_{\mathrm{III}}/d\rho = 0$, $d^4v_{\mathrm{III}}/d\rho^4 = 0$, $d^5v_{\mathrm{III}}/d\rho^5 = 1$ at $\rho = \eta$.

It is easily seen that there are six initial conditions at $\rho = \eta$ for each of the functions v_{I}, v_{II}, and v_{III}. Consequently, with $\rho = \eta$ as a starting point, any one of several numerical forward-integration schemes can be employed to compute these functions. The Runge–Kutta method was used in this investigation inasmuch as it is a part of the program library of the Control Data 1604 computer.

In general, none of the functions v_{I}, v_{II}, and v_{III} will satisfy the boundary conditions (6*b*) which apply at $\rho = 1$. In recognition of this, one defines

$$f = (v)_{\rho=1}, \quad g = \left(\frac{d^2v}{d\rho^2} + \frac{dv}{d\rho}\right)_{\rho=1}, \quad h = \left[\frac{d^3v}{d\rho^3} - (3+\Lambda^2)\frac{dv}{d\rho}\right]_{\rho=1}, \qquad (7a,b,c)$$

so that a solution is achieved when $f = g = h = 0$. Although the f, g, and h for the trial solutions v_{I}, v_{II}, and v_{III} are not necessarily zero, one may seek linear combinations of these which do sum to zero. Thus,

$$c_1 f_{\mathrm{I}} + c_2 f_{\mathrm{II}} + c_3 f_{\mathrm{III}} = 0, \qquad (8a)$$

$$c_1 g_{\mathrm{I}} + c_2 g_{\mathrm{II}} + c_3 g_{\mathrm{III}} = 0, \qquad (8b)$$

$$c_1 h_{\mathrm{I}} + c_2 h_{\mathrm{II}} + c_3 h_{\mathrm{III}} = 0, \qquad (8c)$$

and, in order that these linear algebraic equations be solvable non-trivially, it is necessary that the determinant of the coefficients be zero. Correspondingly, one defines

$$\Delta = \begin{vmatrix} f_{\mathrm{I}} & f_{\mathrm{II}} & f_{\mathrm{III}} \\ g_{\mathrm{I}} & g_{\mathrm{II}} & g_{\mathrm{III}} \\ h_{\mathrm{I}} & h_{\mathrm{II}} & h_{\mathrm{III}} \end{vmatrix}. \qquad (9)$$

Thus, if $\Delta = 0$, one can solve equations (8) and satisfy the boundary conditions at $\rho = 1$. It can be proved rigorously that the foregoing method provides a valid solution of the problem defined by equations (4) and (6); the authors will be pleased to supply such a proof upon request.

However, for prescribed values of μ, η, and Λ, it is not to be expected that any arbitrary trial value of T^* will yield $\Delta = 0$. Rather, one repeats the foregoing operations for a range of trial values of T^*, and in this way there is generated a function $\Delta = \Delta(T^*)$ corresponding to the fixed values of μ, η, and Λ. The value of T^* at which a curve of Δ *vs* T^* crosses the $\Delta = 0$ axis is the Taylor number marking the onset of instability for the given μ, η, and Λ.

Next, one repeats the foregoing operations for another value of Λ, holding the μ and η fixed. This results in another value of T^* which satisfies the differential equation (4) and boundary conditions (6). Further, one may assign a succession of Λ values and find the corresponding T^* values. As expected, for a given μ and η, there exists a Λ at which the T^* is a minimum. This is taken to be the critical Taylor number for the given μ and η.

The decisions involved in the foregoing solution method (e.g. whether $\Delta > 0$ or $\Delta < 0$) lend themselves readily to computer logic. Consequently, the determination of the critical Taylor number can be carried out almost completely within the computer.

Critical Taylor numbers have been thus determined for radius ratios $\eta = R_1/R_2$ of 0·95, 0·75, 0·5, 0·35, 0·25, 0·15, and 0·10 over a wide range of speed ratios $\mu = \Omega_2/\Omega_1$. For $\mu \geqslant 0$ (cylinders rotating in the same direction), computations were carried out without difficulty over the entire range from $\mu = 0$ to $\mu = (R_1/R_2)^2$, the latter value being the limit beyond which the flow is stable. For $\mu < 0$ (cylinders rotating in opposite directions), the computations were extended to speed ratios whose absolute magnitudes were substantially larger than those for $\mu > 0$. The largest negative value of μ for which computations were carried out for each radius ratio was limited by numerical difficulties which could be detected when the curve of Δ *vs* T^* was not smooth. These numerical difficulties are believed to be attributable to loss of significant figures. This condition could be remedied by the use of multiple-precision arithmetic if one wished to consider larger negative speed ratios.

The results of the analysis and their relationship to prior work will be described in the next section.

3. Instability criteria and related results

The results which have been derived by the application of the foregoing solution method are listed in tables 1 and 2, respectively for the cases $\Omega_2/\Omega_1 \geqslant 0$ and $\Omega_2/\Omega_1 < 0$. In addition, a graphical representation of the instability criteria is presented in figures 1, 2, and 3.

Attention may first be given to the general structure of the tables. In each table, the listing is arranged so that information pertaining to a given radius ratio R_1/R_2 is grouped together. In addition to the Ω_2/Ω_1 value which characterizes each case, there is also listed the quotient $\mu/\eta^2 = (\Omega_2/\Omega_1)(R_2/R_1)^2$. The latter has particular relevance inasmuch as the flow is known to be stable when

$(\Omega_2/\Omega_1)(R_2/R_1)^2 > 1$ (Rayleigh 1920). The predicted onset of instability at a given radius ratio and speed ratio occurs at the corresponding T^* value given in the tables.

R_1/R_2	Ω_2/Ω_1	μ/η^2	$\Lambda(1-\eta)$	T^*†	T_T	T_{M+}
0·95	0·0000	0·000	$\sim\pi$	1·517 (7)	1·0003	0·9856
	0·2256	0·250	↓	1·637 (7)	0·9988	0·9921
	0·4500	0·499		2·055 (7)	0·9982	0·9952
	0·6769	0·750		3·545 (7)	0·9979	0·9967
	0·8500	0·942		1·376 (8)	0·9978	0·9974
0·75	0·0000	0·000	$\sim\pi$	2·093 (5)	0·9764	0·9648
	0·1406	0·250	↓	2·324 (5)	0·9832	0·9781
	0·2800	0·498		2·974 (5)	0·9886	0·9865
	0·4219	0·750		5·213 (5)	0·9929	0·9924
	0·5300	0·942		2·055 (6)	0·9957	0·9956
0·50	0·0000	0·000	3·163	7·439 (4)	0·6811	0·9052
	0·0600	0·240	3·150	8·531 (4)	0·7361	0·9310
	0·1250	0·500	3·150	1·137 (5)	0·7862	0·9516
	0·1800	0·720	3·145	1·839 (5)	0·8216	0·9649
	0·2350	0·940	3·145	7·838 (5)	0·8516	0·9758
0·35	0·0000	0·000	3·205	9·464 (4)	−0·7091	0·7653
	0·0306	0·250	3·192	1·123 (5)	−0·5093	0·8108
	0·0612	0·500	3·185	1·518 (5)	−0·4198	0·8468
	0·0918	0·750	3·179	2·754 (5)	−2·073	0·8759
	0·1160	0·947	3·172	1·213 (6)	0·4005	0·8952
0·25	0·0000	0·000	3·240	1·764 (5)	−4·200	0·4347
	0·0156	0·250	3·225	2·134 (5)	−3·632	0·5116
	0·0310	0·496	3·225	2·911 (5)	−3·164	0·5745
	0·0469	0·750	3·225	5·407 (5)	−2·755	0·6289
	0·0600	0·960	3·210	3·164 (6)	−2·463	0·6675
0·15	0·0000	0·000	3·310	6·790 (5)	−25·72	−1·036
	0·0056	0·250	3·300	8·385 (5)	−23·59	−0·8548
	0·0113	0·500	3·280	1·171 (6)	−21·74	−0·6982
	0·0169	0·750	3·270	2·189 (6)	−20·12	−0·5617
	0·0211	0·940	3·260	8·628 (6)	−19·03	−0·4694
0·10	0·0000	0·000	3·360	2·460 (6)	−92·75	−4·329
	0·0025	0·250	3·350	3·070 (6)	−86·51	−3·953
	0·0050	0·500	3·340	4·326 (6)	−80·99	−3·620
	0·0075	0·750	3·330	8·155 (6)	−76·06	−3·324
	0·0095	0·950	3·310	3·898 (7)	−72·54	−3·112

† All T^* values are multiplied by 10^n, where n is the number in parentheses.

TABLE 1. Stability criteria for $\Omega_2/\Omega_1 \geqslant 0$.

In addition to T^*, numerical values of two other Taylor numbers are included in the tables. That one which is denoted by T_T is defined as

$$T_T = T^*\left\{\frac{2\eta^2(1-\eta)^3(1-\mu\eta^{-2})}{(1+\eta)}\left[\frac{\chi^2+(0\cdot00056/0\cdot0571)}{(1712\chi)/(1-\mu)}\right]\right\}, \tag{10a}$$

where

$$\chi = \frac{1+\mu}{1-\mu}-\frac{0\cdot652(1-\eta)}{\eta} \tag{10b}$$

R_1/R_2	$-\Omega_2/\Omega_1$	$-\mu/\eta^2$	$1.546 \times (\eta_0-\eta)\ \Lambda$	T^*†	T_T	T_{M-}
0·95	0·1	0·1108	4·38	1·508 (7)	1·002	2·140
	0·2	0·2216	4·01	1·564 (7)	1·004	1·675
	0·5	0·5540	3·41	1·963 (7)	1·026	1·054
	0·6	0·6648	3·29	2·257 (7)	1·061	0·993
	1·0	1·108	$\sim \pi$	4·588 (7)	−4·14	1·017
	1·5	1·662	↓	9·122 (7)	−11·3	1·023
	2·0	2·216		1·594 (8)	−41·0	1·026
	3·0	3·324		3·834 (8)	−248	1·030
0·75	0·1	0·1778	4·23	2·069 (5)	0·971	1·647
	0·2	0·3556	3·81	2·143 (5)	0·964	1·239
	0·4	0·7111	3·27	2·701 (5)	0·978	0·900
	0·5	0·8889	2·99	3·290 (5)	1·097	0·862
	1·0	1·778	3·06	9·500 (5)	−8·13	0·937
	1·5	2·667	3·08	2·020 (6)	−47·9	0·957
	2·0	3·556	3·14	3·697 (6)	−170	0·972
0·50	0·1	0·400	3·82	7·007 (4)	0·591	1·048
	0·2	0·800	3·20	7·846 (4)	2·82	0·776
	0·3	1·200	3·08	1·048 (5)	−1·46	0·726
	0·35	1·400	2·98	1·262 (5)	−2·27	0·743
	0·5	2·000	3·06	2·088 (5)	−7·99	0·795
	1·0	4·000	3·07	6·772 (5)	−109·9	0·857
	1·2	4·800	3·08	9·761 (5)	−229·2	0·875
	1·5	6·000	3·09	1·55 (6)	−570	0·885
0·35	0·05	0·408	3·95	8·381 (4)	−1·22	0·946
	0·10	0·816	3·34	8·536 (4)	−2·07	0·718
	0·17	1·388	3·09	1·077 (5)	−4·56	0·634
	0·20	1·633	2·95	1·253 (5)	−6·48	0·644
	0·35	2·857	3·03	2·491 (5)	−28·0	0·710
	0·50	4·082	3·04	4·220 (5)	−83·2	0·746
	0·75	6·122	3·04	8·552 (5)	−334	0·791
	0·85	6·939	3·04	1·082 (6)	−527	0·802
0·25	0·05	0·800	3·51	1·435 (5)	−7·09	0·685
	0·075	1·200	3·08	1·503 (5)	−9·67	0·597
	0·10	1·600	2·95	1·698 (5)	−13·7	0·571
	0·15	2·400	2·96	2·417 (5)	−28·5	0·605
	0·20	3·200	2·96	3·268 (5)	−52·6	0·632
	0·35	5·600	2·97	6·670 (5)	−217	0·683
	0·50	8·000	2·98	1·168 (6)	−633	0·722
0·15	0·025	1·111	3·39	4·884 (5)	−40·7	0·572
	0·040	1·778	3·18	5·151 (5)	−58·0	0·511
	0·050	2·222	3·04	5·666 (5)	−75·2	0·509
	0·100	4·444	3·03	9·334 (5)	−226	0·555
	0·150	6·666	3·02	1·364 (6)	−500	0·583
	0·200	8·888	2·99	1·883 (6)	−951	0·608
	0·250	11·11	2·99	2·497 (6)	−1644	0·629
	0·300	13·33	2·98	3·264 (6)	−2697	0·658
0·10	0·010	1·000	3·61	1·673 (6)	−128	0·577
	0·016	1·600	3·11	1·613 (6)	−162	0·502
	0·025	2·500	2·98	1·759 (6)	−240	0·480
	0·050	5·000	3·05	2·520 (6)	−610	0·511
	0·075	7·500	3·02	3·310 (6)	−1173	0·530
	0·090	9·000	3·01	3·827 (6)	−1626	0·540
	0·120	12·000	2·99	4·963 (6)	−2844	0·559
	0·150	15·000	2·98	6·206 (6)	−4536	0·574

† All T^* values are multiplied by 10^n, where n is the number in parentheses.

TABLE 2. Stability criteria for $\Omega_2/\Omega_1 < 0$.

The reason for considering T_T is the fact that Taylor predicted the instability criterion $T_T = 1$ on the basis of his analysis for the case 'when $(R_2-R_1)/R_1$ is small, but not neglected'. The Taylor number denoted by T_{M+} in table 1 is given by

$$T_{M+} = T^*\left\{\frac{2\eta^2(1-\eta)^3(1-\mu\eta^{-2})}{1712(1+\eta)}\left[(1+\mu)-\frac{3}{2}\left(\frac{1-\eta}{1+\eta}\right)(1-\mu)\right]\right\}. \tag{11}$$

The analysis of Meksyn for the case of $\mu \geqslant 0$ leads to the criterion $T_{M+} = 1$.

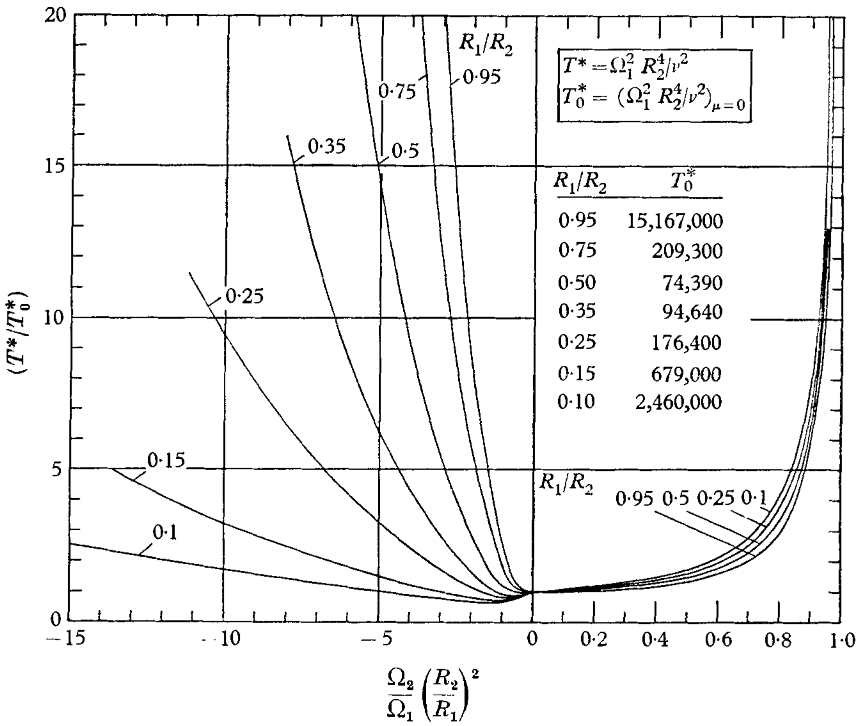

FIGURE 1. Conditions for instability of the flow between two rotating cylinders.

Finally, the definition of the T_{M-} of table 2 is

$$T_{M-} = T^*\left[\frac{2(1-\eta)^5(1-\mu\eta^{-2})^2}{283\,\eta_0(\eta^{-2}-1)^2}\right], \tag{12a}$$

wherein η_0 represents the radial position at which the main-flow velocity V_θ (equation (1)) vanishes when $\mu < 0$, i.e.

$$\eta_0 = \frac{R_0}{R_2} = \left[\frac{1-\mu}{1-\mu\eta^{-2}}\right]^{\frac{1}{2}}. \tag{12b}$$

Meksyn's analysis for $\mu < 0$ predicts the criterion $T_{M-} = 1$.

In addition to providing a convenient comparison with the predictions of Taylor and Meksyn, the listing of the T_T and T_M values facilitates interpolation of the results for cases other than those tabulated. In particular, it is readily seen that the T_{M+} is a slowly and smoothly varying function of Ω_2/Ω_1 at a given radius ratio. A similar statement applies to T_{M-} for Ω_2/Ω_1 values that are not too close to zero.

The values of the dimensionless wave-numbers Λ corresponding to the onset of instability are also presented in tables 1 and 2. For the case of $\mu \geqslant 0$ (table 1) the grouping $\Lambda(1-\eta)$ is listed in recognition of the fact that $\Lambda(1-\eta) = \pi$ on the

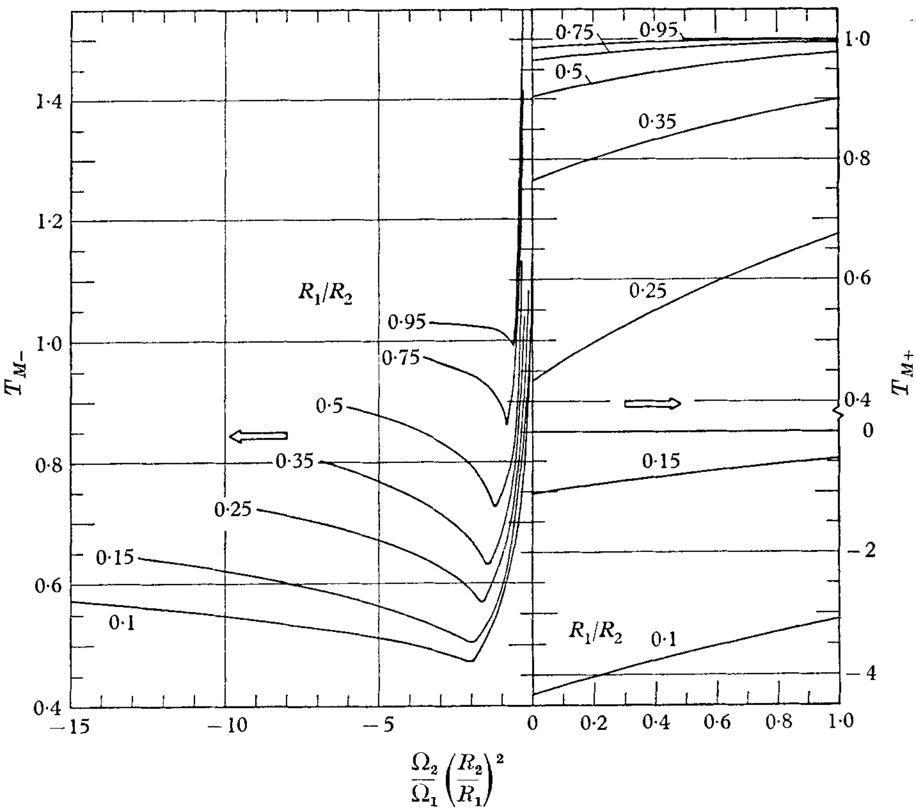

FIGURE 2. Instability results in terms of Meksyn's grouping T_M.

basis of the narrow-gap analysis. On the other hand, for $\mu < 0$ (table 2), there is listed the grouping $1{\cdot}546\,(\eta_0-\eta)\,\Lambda$ whose value is predicted to be π by Meksyn's asymptotic analysis.

The T^* results may be discussed with the aid of figure 1. The ordinate of the figure is the ratio of the T^* which denotes instability at a given μ and η to the T^* which denotes instability when $\mu = 0$ at the same η. This information is plotted against $(\Omega_2/\Omega_1)(R_2/R_1)^2$, and the curves are parameterized by R_1/R_2. There is a change in the abscissa scale at $\mu = 0$. The T^* values corresponding to $\mu = 0$ are denoted as T_0^* and are tabulated in the figure.

With certain exceptions to be noted below, the curves are seen to rise monotonically with increasing values of $|\Omega_2/\Omega_1|$, thereby indicating that larger values of Ω_1 are required to bring about instability. For $\Omega_2/\Omega_1 \geqslant 0$, the curves for the various radius ratios are contained in a tight bundle and approach infinity asymptotically as $(\Omega_2/\Omega_1)(R_2/R_1)^2$ approaches unity. On the other hand, for

$\Omega_2/\Omega_1 < 0$, there is a marked spreading of the curves as a function of radius ratio.

Further inspection of the figure reveals that the minima in the stability curves do not lie precisely at $\Omega_2/\Omega_1 = 0$; rather, the minima are achieved at small negative values of Ω_2/Ω_1. The departure of the minimum T^* from T_0^* is accentuated as R_1/R_2 decreases. The existence of such off-axis minima has been previously

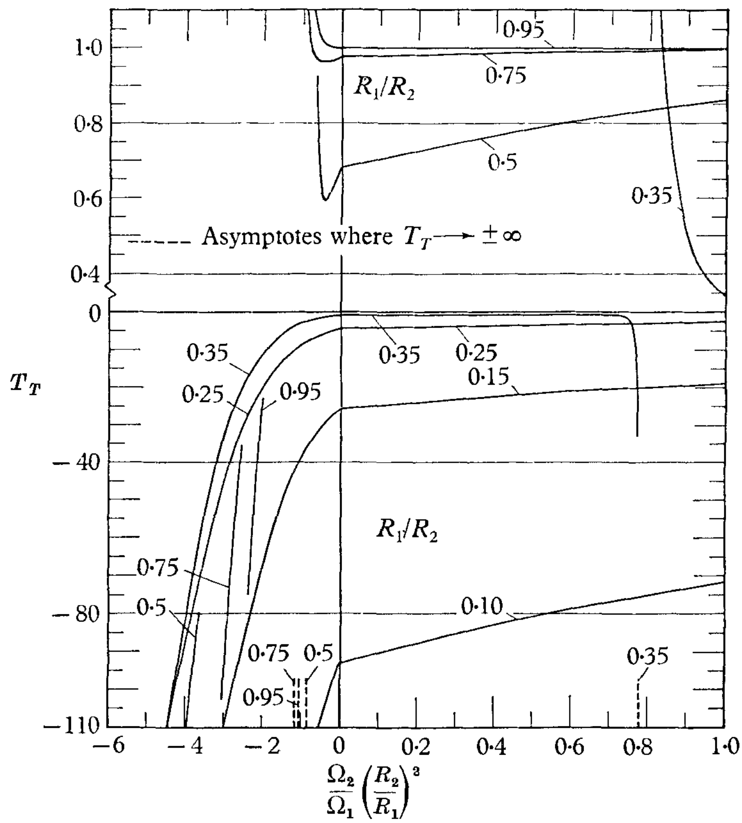

FIGURE 3. Instability results in terms of Taylor's grouping T_F.

noted by Donnelly & Fultz (1960), both in connexion with stability diagrams based on their own experimental data and on the calculations of Chandrasekhar.

The discussion of results for the Taylor numbers T_M and T_T is facilitated by reference to figures 2 and 3, respectively. Consideration will first be given to T_M, the definition of which is stated for the cases $\mu \geqslant 0$ and $\mu < 0$ in equations (11) and (12), respectively. Inspection of the plotted results for T_{M+} indicates that Meksyn's prediction of $T_{M+} = 1$ for the onset of instability is reasonably accurate for moderately small radius ratios. Furthermore, the Meksyn prediction takes on an even more favourable appearance when it is realized that the instability value of Ω_1 is computed from $T_{M+}^{\frac{1}{2}}$. Indeed, it would appear that the relationship $T_{M+} = 1$ could be employed with confidence for $R_1/R_2 \geqslant 0{\cdot}5$ and could serve as a good first approximation for computing Ω_1 for R_1/R_2 as low as 0·35. As R_1/R_2 decreases, T_{M+} departs more and more from unity and finally goes negative.

However, as is evident from figure 2, T_{M+} is a smooth function of Ω_2/Ω_1 for any given radius ratio, and this facilitates accurate interpolation to find results at values of Ω_2/Ω_1 other than those tabulated here.

Turning next to the T_{M-} results of figure 2, it is seen that the relationship $T_{M-} = 1$ is strongly violated for values of Ω_2/Ω_1 near zero. This state of affairs was foreseen by Meksyn. For the larger radius ratios, the Meksyn prediction ($T_{M-} = 1$) is closely approached at larger negative values of Ω_2/Ω_1. However, for the smaller radius ratios, T_{M-} falls well below unity even at the larger $|\Omega_2/\Omega_1|$. It may be noted from the figure that, for Ω_2/Ω_1 values which lie to the left of their respective minima, the curves are smooth and slowly varying. Consequently, in this range, they provide an accurate means of interpolation to find results for cases other than those listed in table 2.

Attention may now be given to the graphical presentation for the Taylor number T_T, figure 3. Inspection of this figure reveals that Taylor's prediction ($T_T = 1$) holds only for radius ratios near unity, and then only for values of Ω_2/Ω_1 which are positive or perhaps slightly negative. At smaller radius ratios, T_T takes on negative values which can be quite large. A similar statement applies at larger negative values of Ω_2/Ω_1 even when R_1/R_2 is near unity. The decisive term in establishing the sign of T_T is the χ factor in equation (10*b*). It is clear from Taylor's presentation that he was aware of the limitations of the prediction $T_T = 1$. The information given in tables 1 and 2 and in figure 3 is useful in setting quantitative limits for the conditions under which this prediction can be applied.

Another quantity of interest is the wave-number $\Lambda = \lambda R_2$ at which the onset of instability occurs. For the case $\mu \geqslant 0$, it is seen from table 1 that the relationship $\Lambda(1-\eta) = \pi$ provides a very good correlation of the results for the entire range of radius ratios studied here. For $R_1/R_2 \geqslant 0{\cdot}35$, this relationship holds to within 2 % or better. The greatest deviation is about 6·6 % and this occurs for $R_1/R_2 = 0{\cdot}1$.

For the case $\mu < 0$, the wave-number results listed in table 2 may be compared with the Meksyn prediction $1{\cdot}546\ (\eta_0-\eta) = \pi$. It is seen that this relationship is quite successful in correlating Λ values for the larger negative values of Ω_2/Ω_1. For the range of negative Ω_2/Ω_1 in which the foregoing correlation is inapplicable (i.e. for Ω_2/Ω_1 near zero), one may use the relationship $\Lambda(1-\eta) = \pi$. It may be noted, however, that the latter leads to erroneous values of Λ at large negative Ω_2/Ω_1.

It is also of interest to compare the present instability results for the wide-gap case with those of other investigators. Chandrasekhar & Elbert (1962) provide tabular results for $R_1/R_2 = 0{\cdot}5$ in the range $-0{\cdot}5 \leqslant \mu < (R_1/R_2)^2$. Upon expressing their instability results in terms of the T^* parameter and comparing with tables 1 and 2, one finds excellent agreement. Results for $R_1/R_2 = 0{\cdot}5$ are presented graphically by Kirchgässner (1961) for $-0{\cdot}4 \leqslant \mu < (R_1/R_2)^2$. These are also in satisfactory agreement with those of this investigation.

Finally, it is appropriate to compare the present instability predictions with those of experiment. Recently, Donnelly & Fultz (1960) have reported an extensive investigation of the case $R_1/R_2 = 0{\cdot}5$. These authors have compared their data with the predictions of Chandrasekhar in the range $-0{\cdot}5 \leqslant \mu \leqslant 0{\cdot}186$

and found very good agreement. It is only necessary, therefore, to give further consideration to the range of larger negative Ω_2/Ω_1. For $-1{\cdot}13 \leqslant \mu \leqslant -0{\cdot}5$, the agreement between analysis and experiment continues to be very good. However, for $\mu < -1{\cdot}13$, the data appear to be somewhat scattered. For instance, for the successive values of $-\mu = 1{\cdot}164$, 1·178, 1·283, 1·291, 1·327, and 1·339, the corresponding values of $T^* \times 10^{-6}$ computed from the data are 1·21, 1·03, 1·04, 1·52, 1·24, 1·08. The presence of such scatter in the data discourages comparison of analysis and experiment in this range.

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